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# Nested Krylov methods for shifted linear systems arising in frequency-domain wave simulations

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## Motivation

We consider the time-harmonic elastic wave equation at multiple frequencies $\omega_k$,

$$-\omega_k^2 \rho(\mathbf{x})\mathbf{u} - \nabla \cdot \tau(\mathbf{u}) = \mathbf{s}, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^{2,3}. \tag{1}$$

Discretization of $(1)$ including Sommerfeld boundary conditions leads to

$$\left(K + i\omega_k C - \omega_k^2 M\right) \underline{\mathbf{u}} = \underline{\mathbf{s}},$$

which can be re-arranged to

$$\left[\begin{pmatrix} iM^{-1}C & M^{-1}K \\ I & 0 \end{pmatrix} - \omega_k \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right] \begin{pmatrix} \omega_k \underline{\mathbf{u}} \\ \underline{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} M^{-1}\underline{\mathbf{s}} \\ 0 \end{pmatrix}. \tag{2}$$

## Multi-shift Krylov methods

Shifted linear systems like $(2)$ are of the form

$$(A - \omega_k I)\mathbf{x}_k = \mathbf{b}, \quad k = 1, ..., N_\omega. \tag{3}$$

Idea for simultaneous solution: Krylov subspaces are shift-invariant, i.e.

$$\mathcal{K}_m(A, \mathbf{b}) \equiv \text{span}\{\mathbf{b}, A\mathbf{b}, ..., A^{m-1}\mathbf{b}\} = \mathcal{K}_m(A - \omega I, \mathbf{b}), \quad \text{for all } \omega \in \mathbb{C}.$$

## Nested preconditioners for shifted problems

Preserve shift-invariance when preconditioning $(3)$:

$$\mathcal{K}_m(A\mathcal{P}^{-1}, \mathbf{b}) = \mathcal{K}_m((A - \omega I)\mathcal{P}_\omega^{-1}, \mathbf{b}). \tag{4}$$

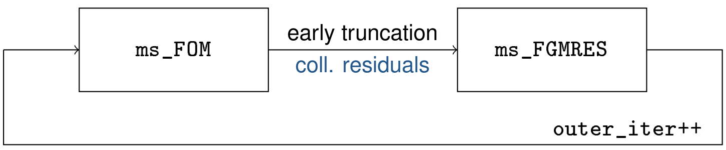
In [2], polynomial preconditioners of degree $n$ are introduced,

$$\mathcal{P}^{-1} \equiv \sum_{i=1}^{n} \alpha_i A^i \approx A^{-1}, \quad \mathcal{P}_\omega^{-1} \equiv \sum_{i=1}^{n} \alpha_i^{(\omega)} A^i \approx (A - \omega I)^{-1},$$

that preserve shift-invariance in $(4)$.

**Nested Krylov preconditioner [1]:**

- Use a Krylov polynomial as an inner preconditioner, e.g. `ms_FOM`.
- Truncation of inner method at `tol` $\sim 0.1$, see Fig. 3.
- Use preconditioner in a flexible outer Krylov iteration, e.g. `ms_FGMRES`.



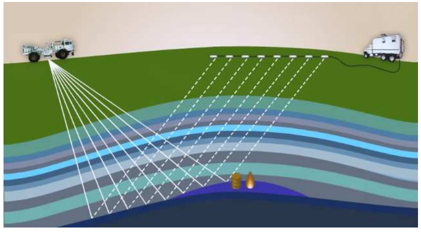



Fig. 1: Full-waveform inversion.

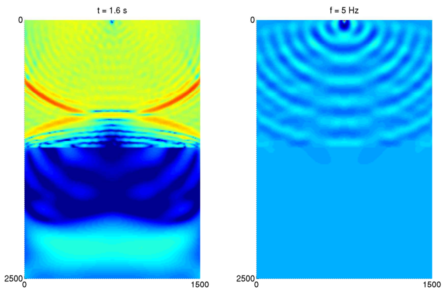



Fig. 2: Solution of $(1)$ for a two-layered test case in time-domain (left) and frequency-domain (right).

## Results

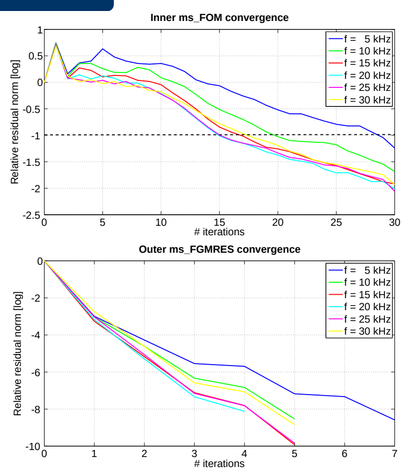



Fig. 3: Convergence curves of nested `FOM-FGMRES` for $N_\omega = 6$ frequencies.

## References

[1] M. Baumann, M. B. van Gijzen. *Nested Krylov methods for shifted linear systems.* Technical report 14-01, Delft University of Technology, 2014.

[2] M. I. Ahmad, D. B. Szyld, M. B. van Gijzen. *Preconditioned multishift BiCG for $\mathcal{H}_2$-optimal model reduction.* Report 12-06-15, Temple University, 2013.