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Fast iterative solution of the time-harmonic elastic wave equation at multiple frequencies

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Application

Geo-scientists want to analyze the earth interior:

- The earth interior consists of several layers with different physical properties,
- Seismic exploration: send sound waves into the earth and analyze their reflection behavior,
- The properties of an oil reservoir can be derived by matching experimental and numerical results within an optimization loop.

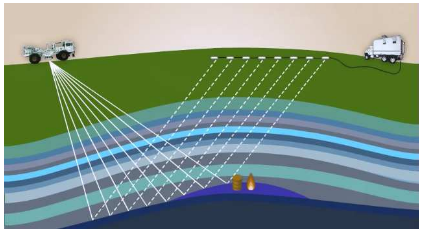

Fig. 1: Wave propagation through the earth

The mathematical model

Time-harmonic elastic wave equation: For many angular frequencies ω_j, we assume as a test problem a homogeneous medium and aim to solve:

$$\begin{aligned} -\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) - \omega_j^2\rho_s\mathbf{u} &= \mathbf{r}, \quad \text{in } \Omega\subset\mathbb{R}^D, \\ i\gamma\omega_j\rho_s B\mathbf{u} + \left[\lambda(\nabla\cdot\mathbf{u}) + \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\right]\mathbf{n} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where

- $\mathbf{u}\in\mathbb{R}^D$ is the displacement vector in two ($D=2$) or three ($D=3$) dimensions,
- μ, λ are the so-called Lamé parameters,
- $\rho_s\in\mathbb{R}$ is the (constant) material density,
- B is defined componentwise, $B_{i,j}\equiv c_p n_i n_j + c_s t_i t_j$, for $D=2$.

Spatial discretization: After FEM-discretization, we obtain the linear systems

$$(K + i\omega_j C - \omega_j^2 M)\underline{\mathbf{u}} = \underline{\mathbf{r}}, \tag{2}$$

with K, C, M being symmetric and sparse. Here, C contains the boundary conditions.

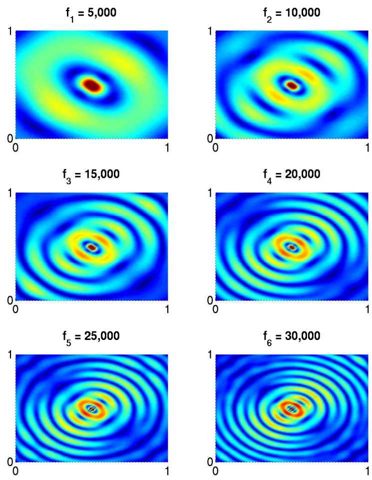

Fig. 2: Solution of (1) on $\Omega=[0,1]\times[0,1]$ with point source $\mathbf{r}\equiv\boldsymbol{\delta}(\mathbf{x}-(0.5,0.5)^T)$ and absorbing boundary conditions ($\gamma=1$)

Our approach

Shifted linear systems: Note that we can re-write (2) as:

$$\left[\begin{pmatrix} iM^{-1}C & M^{-1}K \\ I & 0 \end{pmatrix} - \omega_j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\begin{pmatrix} \omega_j\underline{\mathbf{u}} \\ \underline{\mathbf{u}} \end{pmatrix} = \begin{pmatrix} M^{-1}\underline{\mathbf{r}} \\ 0 \end{pmatrix}, \tag{3}$$

which is of the form

$$(A-\omega_j I)\mathbf{x}_j = \mathbf{b}, \quad j=1,...,N. \tag{4}$$

Idea for simultaneous solve: Krylov subspaces are shift-invariant, i.e.

$$\mathcal{K}_i(b,A)\equiv \text{span}\,\{b, Ab, ..., A^{i-1}b\} = \mathcal{K}_i(b, A-\omega I), \quad \text{for all } \omega\in\mathbb{R}.$$

Future work:

- Implement a more realistic test case for $D=3$ and inhomogeneous material,
- Improve preconditioners for shifted Krylov methods,
- Consider multiple right-hand sides in (4) as well.

Results

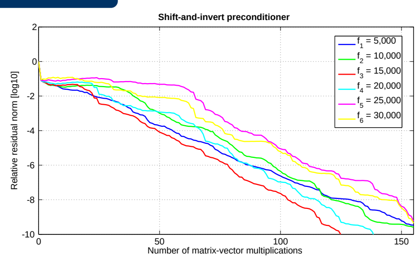

Fig. 3: Convergence of multi-shift IDR(1) was obtained $\sim$ 3.7 times faster than the consecutive solution of (3)

References

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