Fast Iterative Solution of the Time-Harmonic Elastic Wave Equation at Multiple Frequencies

Manuel M. Baumann

January 10, 2018

Question you have asked me today...

- Are you nervous? → Yes!

Questions you have asked me during the last years...

- What is your PhD project about?
- What is numerical linear algebra?
- What have you been doing all day?
  (The German word for this is: *rumdoktorn*)

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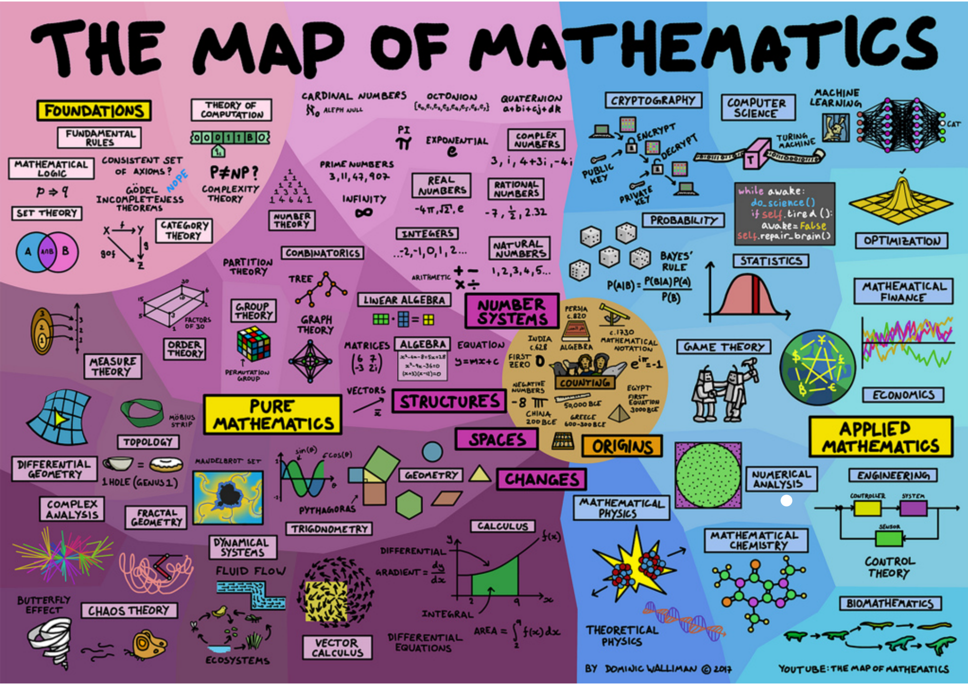
THE MAP OF MATHEMATICS
FOUNDATIONS
FUNDAMENTAL RULES
MATHEMATICAL LOGIC
p ⇒ q
SET THEORY
A
A∩B
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THEORY OF COMPUTATION
CONSISTENT SET OF AXIOMS?
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X
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3, i, 4+3i, -4i
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NUMBER THEORY
INTEGERS
..-2, -1, 0, 1, 2...
NATURAL NUMBERS
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+ − × ÷
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MÖBIUS STRIP
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1 HOLE (GENUS 1)
MANDELBROT SET
sin(θ)
cos(θ)
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AREA = ∫ f(x) dx
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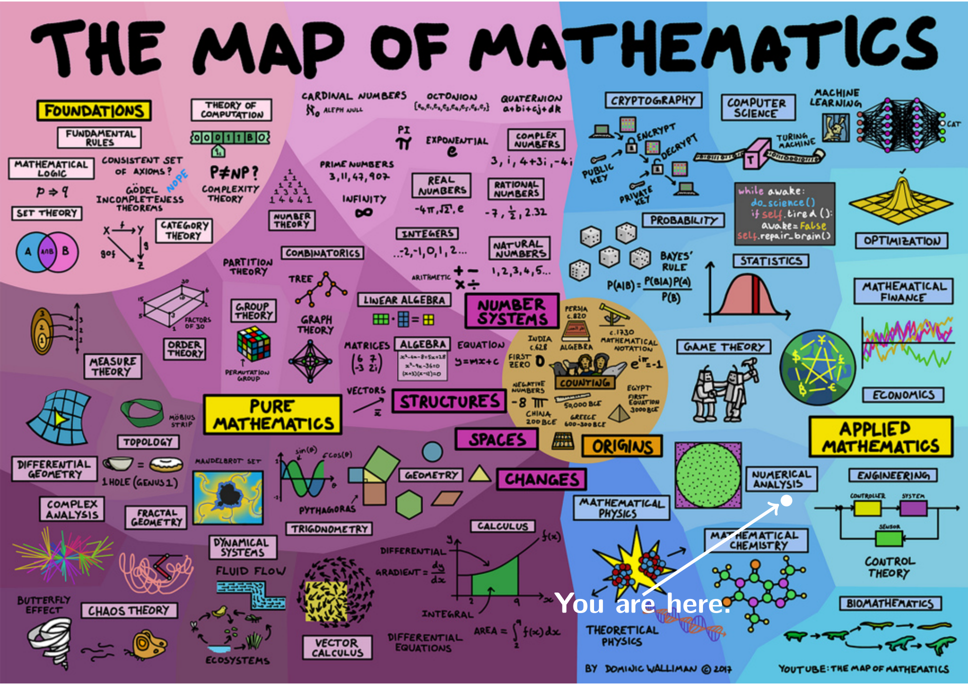
YOUTUBE: THE MAP OF MATHEMATICS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What is applied mathematics?

"Applied maths is about using mathematics to solve real world problems neither seeking nor avoiding mathematical difficulties."

–Lord Rayleigh

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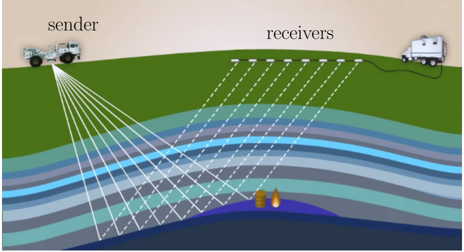

Interplay of...

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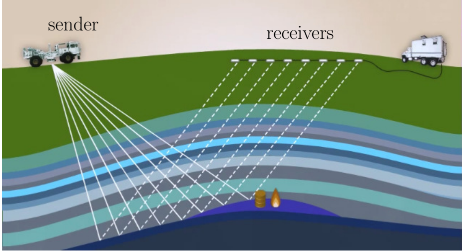

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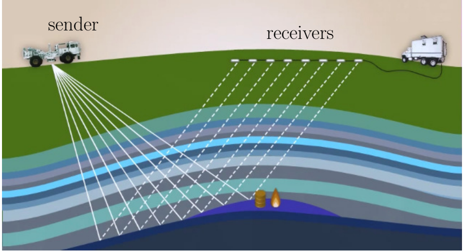

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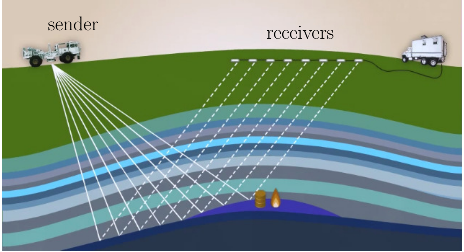

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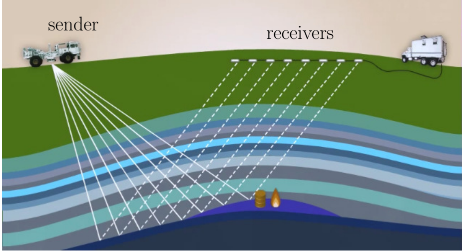

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- measurements,
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- computer simulations
  $\hookrightarrow$ matrix computations

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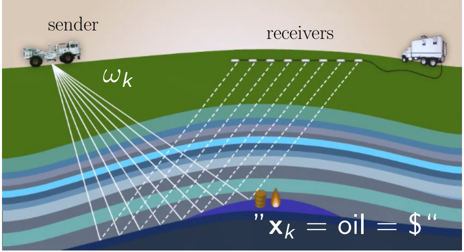

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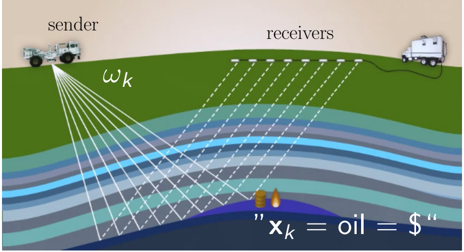
"Solve the linear systems of equations,

$$(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b},$$

efficiently (= fast and at low memory) for multiple frequencies."

Seismic Full-Waveform Inversion



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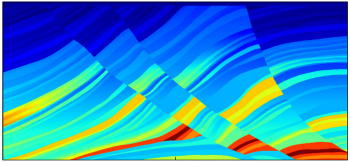
Density distribution



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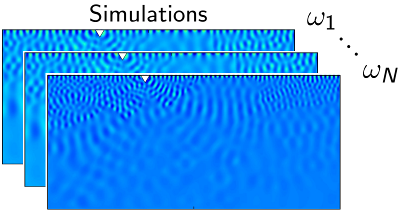

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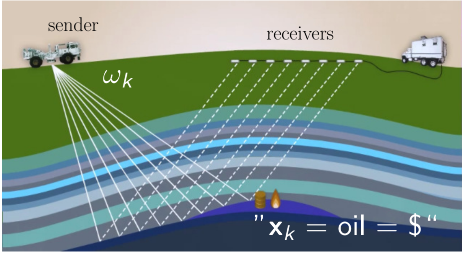

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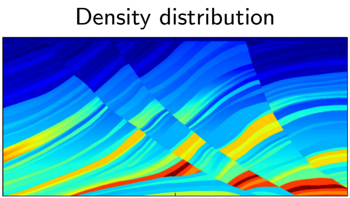

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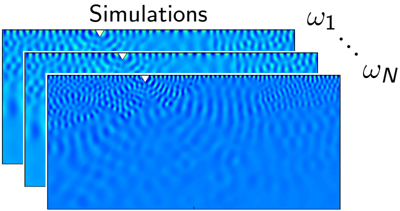

Numerical Linear Algebra

A very classical linear algebra problem,

$$
\begin{aligned}
🚲 + ⚽ \quad &= 1500 \\
🚲 \quad + ☕☕ &= 7.5 \\
⚽⚽ - ☕☕☕ &= 160
\end{aligned}
$$

A more formal way of writing this,

$$
\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}
\begin{bmatrix} 🚲 \\ ⚽ \\ ☕ \end{bmatrix}
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The matrix A is symmetric

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The matrix A is symmetric and sparse.

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sequentially semi-seperable

SPD

Hermitian

indefinite

symmetric

block tri-diagonal

dense

skew-symmetric

The matrix A can be...

rectangular

Port-Hamiltonian

nilpotent

upper Hessenberg

square

sparse

diagonalizable

invertible

low-rank

ill-conditioned

positive (semi-)definite

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Shifted systems vs. matrix equation

Two main approaches for solving,

$$\boxed{(K + i\omega_k C - \omega_k^2 M)\mathbf{x}_k = \mathbf{b}, \quad k > 1.}$$

Shifted systems vs. matrix equation

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Shifted systems

$$\left(\begin{bmatrix} iC & K \\ I & 0 \end{bmatrix} - \omega_k \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} \omega_k \mathbf{x}_k \\ \mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

- Most work for $\mathbf{x}_0$ (at $\omega = 0$)
- Requires preconditioning

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Matrix equation

$$K\mathbf{X} + iC\mathbf{X}\Omega - M\mathbf{X}\Omega^2 = \mathbf{B}$$

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Preconditioning

Let $A := K + i\omega C - \omega^2 M$

Solve large-scale linear system,

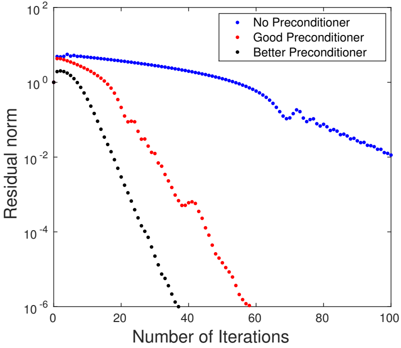
$$A\mathbf{x} = \mathbf{b}, \quad \text{with } A \in \mathbb{C}^{N\times N}, N \gg 1 \qquad (*)$$

with an iterative method, i.e. compute $\mathbf{x}_i$ with $\mathbf{x}_i \to \mathbf{x}$ as $i \to \infty$.

Instead of $(*)$, solve the system

$$P^{-1}A\mathbf{x} = P^{-1}\mathbf{b},$$

where P is a preconditioner.



Preconditioning

However, it's often not that simple!

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Main challenges:

- multiple linear systems
- single preconditioner
- wide frequency range
- preserve structure

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$\tau^* = ?$

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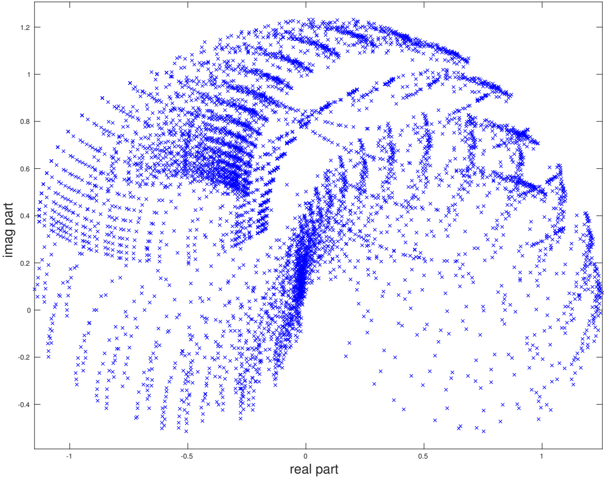

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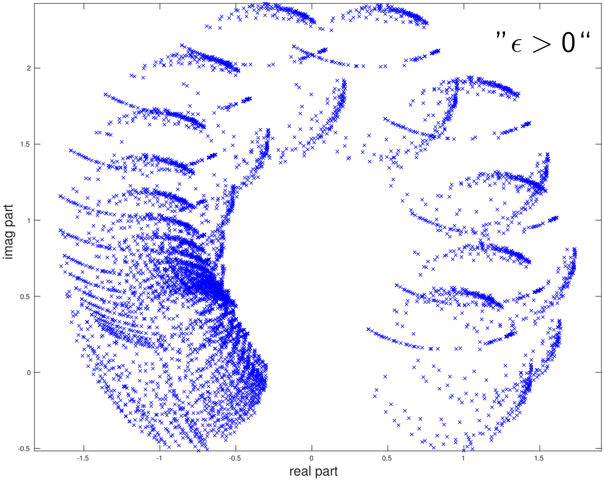

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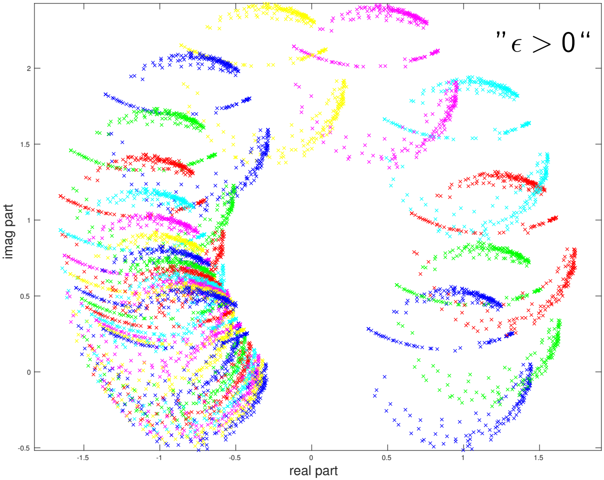

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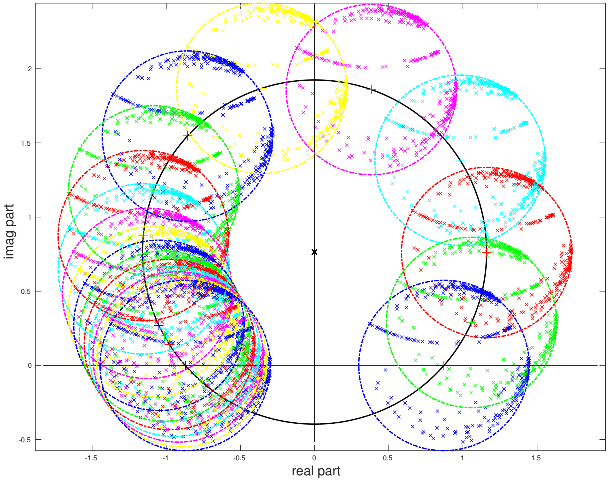
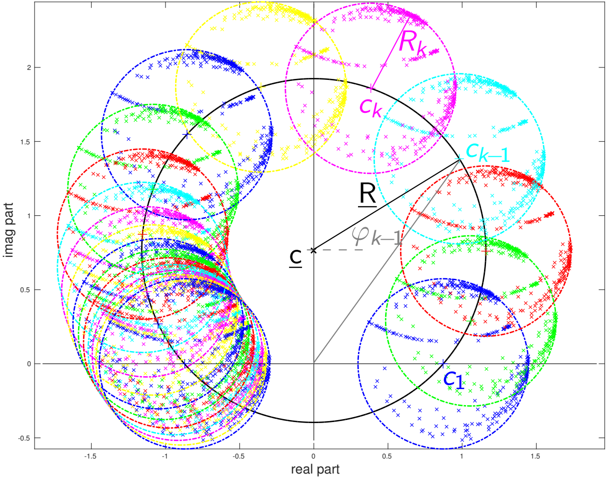


Spectral analysis



TUDelft

Spectral analysis

Thm.: Optimal seed shift for multi-shift GMRES [B/vG, 2016]

(i) For $\lambda \in \mathbf{\Lambda}[\mathcal{A}\mathcal{B}^{-1}]$ it holds $\Im(\lambda) \geq 0$.

(ii) The preconditioned spectra are enclosed by circles of radii R_k and center points c_k.

(iii) The points $\{c_k\}_{k=1}^N \subset \mathbb{C}$ described in statement (ii) lie on a circle with center $\underline{c}$ and radius $\underline{R}$.

(iv) Consider the preconditioner $\mathcal{P}(\tau^*) = \mathcal{A} - \tau^*\mathcal{B}$. An optimal seed frequency τ^* for preconditioned multi-shift GMRES is given by,

$$\tau^*(\epsilon, \omega_1, \omega_N) = \min_{\tau \in \mathbb{C}} \max_{k=1,..,N} \left(\frac{R_k(\tau)}{|c_k|} \right) = \ldots =$$

$$= \frac{2\omega_1\omega_N}{\omega_1 + \omega_N} - i\frac{\sqrt{[\epsilon^2(\omega_1+\omega_N)^2 + (\omega_N - \omega_1)^2]\,\omega_1\omega_N}}{\omega_1 + \omega_N}$$

Spectral analysis

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Spectral analysis

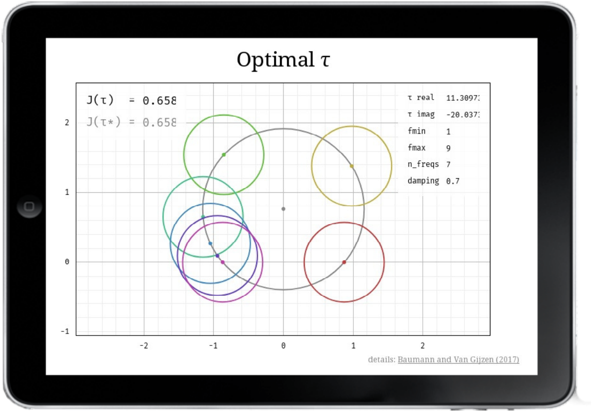
Proof:

Spectral analysis

Proof: Not now.

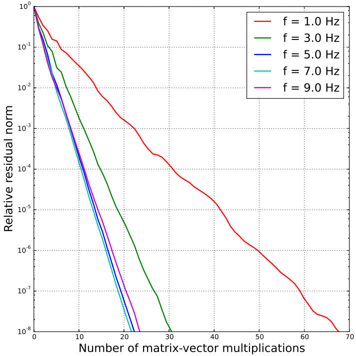
Spectral analysis

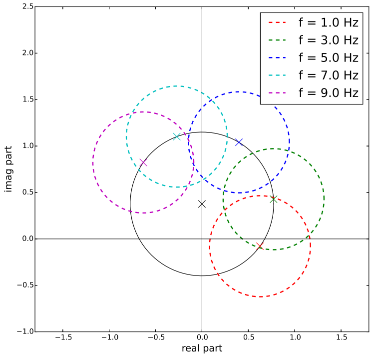
Proof: There is an App for that.



Convergence behavior and spectral bounds

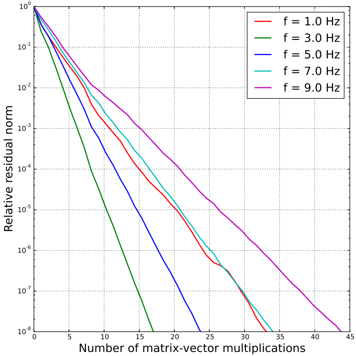
For any τ...

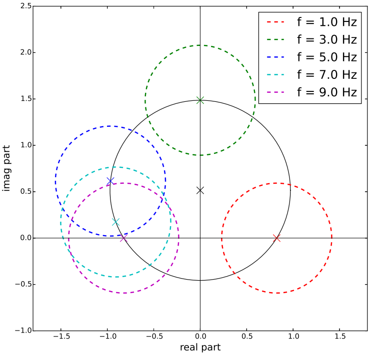




Convergence behavior and spectral bounds

For the optimal τ^*...





Lot's of details...

What happens today?

15:00 – 16:00	Formal PhD defense
16:15 – 17:30	Reception (in this building)
21:00 – ??	*More* reception (borrel) at Prinsenkwartier

$\xrightarrow{?}$


Thank you all for coming!