A generalized POD space-time Galerkin scheme for parameter dependent dynamical systems

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Exemplary Setup

We consider a parameter-dependent PDE

$$i(t, x) = f(x, i(t), i_0), \quad (0, T) \times \Omega, \quad i(0) = i_0 \in Y,$$

and a finite element discretization with the FEM space $Y = \text{span}\{v_1, \ldots, v_N\}$ that leads to

$$M_i y(t) = f(y(t), i_0), \quad (0, T), \quad y(0) = y_0 \in \mathbb{R}^N,$$

where $M_i$ is the mass matrix of $Y$.

Generalized Measurements and POD modes

Fix a $\mu = \mu_k$. Let $S = \text{span}\{v_1, \ldots, v_N\} \subset L^2(0, T)$ and consider the generalized measurement matrix

$$Y_{\text{gen}} \coloneqq \begin{pmatrix} y_1(t_1) & \cdots & y_1(t_N) \\ \vdots & \ddots & \vdots \\ y_N(t_1) & \cdots & y_N(t_N) \end{pmatrix}, \quad \text{cf. } Y_{\text{POD}} \coloneqq \begin{pmatrix} y_1(t_1) \\ \vdots \\ y_N(t_1) \\ \vdots \\ y_1(t_N) \\ \vdots \\ y_N(t_N) \end{pmatrix},$$

the snapshot matrix known from POD.

Generalized spatial POD modes

From the measurement matrix $Y_{\text{gen}}$ we can obtain an optimal (in the sense of Lemma 1) reduced basis $(\psi_1, \ldots, \psi_N)$ for a discretization via

$$\psi_j = V_j^{\text{gen}} v_j,$$

where $V_j$ is the $j$th leading singular vector of $Y_{\text{gen}} M_{\text{gen}}^{-1/2}$.

Generalized time POD modes

With the same arguments we can obtain an optimal reduced basis $(\psi_1, \ldots, \psi_N)$ for the discretization via

$$\psi_j = U_j^{\text{gen}} v_j,$$

where $U_j$ is the $j$th leading singular vector of $M_{\text{gen}}^{1/2} Y_{\text{gen}}^{\text{trans}} M_{\text{gen}}^{-1/2}$.

Adding the parameter dependency

A discretization of the parameter domain with $p$ degrees of freedom adds another dimension to the generalized measurement matrix turning it into a tensor $y \in \mathbb{R}^{p \times N}$,

$$\begin{pmatrix} y(t_1, x_1) & \cdots & y(t_1, x_p) \\ \vdots & \ddots & \vdots \\ y(t_N, x_1) & \cdots & y(t_N, x_p) \end{pmatrix}.$$

Then, optimal bases are obtained via a higher-order SVD, i.e. via SVDs of tensor unfoldings with respect to the space dimension

$$Y^{(1)} \coloneqq \begin{pmatrix} y_1(t_1, x_1) & \cdots & y_1(t_1, x_p) \\ \vdots & \ddots & \vdots \\ y_N(t_1, x_1) & \cdots & y_N(t_1, x_p) \end{pmatrix},$$

and with respect to the time dimension

$$Y^{(2)} \coloneqq \begin{pmatrix} y(t_1, x_1) & \cdots & y(t_1, x_p) \\ \vdots & \ddots & \vdots \\ y(t_N, x_1) & \cdots & y(t_N, x_p) \end{pmatrix},$$

respectively, cf. Lemma 1.

The basic theory

$L^2$ projections onto the measurements

Lemma 1 (See Chapter 3.3 in [8]). The $L^2(0, T)$-orthogonal projection $\tilde{y}(t)$ of the state vector $y(t)$ onto the spanned by the measurements is given as

$$\tilde{y}(t) = Y y_0 M_i^{-1/2} \tilde{v}(t),$$

where $\tilde{v} \coloneqq [v_1, v_2, \ldots, v_N]$, and where $[y_1, \ldots, y_N]$, $[y_1, \ldots, y_N]$. The generalized POD basis can be computed via a truncated SVD of

$$Y_{\text{gen}} M_{\text{gen}}^{-1/2}.$$

Higher order SVDs [8]

For a third-order tensor like $y \in \mathbb{R}^{p \times N \times p}$ there exists a HOSVD

$$y = U_1^{(p)} x_1 U_2^{(p)} x_2 U_3^{(p)} x_3 L^{(p)}$$

with the core tensor $L \in \mathbb{R}^{p \times N \times p}$ satisfying some orthogonality properties and with unitary matrices $U_1^{(p)} \in \mathbb{C}^{p \times p}$, $U_2^{(p)} \in \mathbb{C}^{N \times N}$, and $U_3^{(p)} \in \mathbb{C}^{p \times p}$. Here, $v_1, v_2, \ldots, v_N$ denote tensor-matrix multiplications. We define a matrix unfolding $y_{\text{gen}}(x) \in \mathbb{R}^{p \times p \times N}$ of the tensor $Y$ via putting all elements belonging to the index $x$ into one respective row. Similarly, we define the unfoldings $Y_{\text{gen}}^{(1)}(x) \in \mathbb{R}^{N \times p}$ and $Y_{\text{gen}}^{(2)}(x) \in \mathbb{R}^{p \times N}$.

We can calculate $U_1^{(p)}$, $U_2^{(p)}$, and $U_3^{(p)}$ in (1) by means of three SVDs like $Y_{\text{gen}}^{(1)}(x) = L^{(1)}(x)$, $L^{(2)}(x) = L^{(2)}(x)$, with $L^{(1)}$ diagonal with entries $l_{11} \geq l_{12} \geq \ldots \geq l_{1p} \geq 0$ and $W^{(1)}$ column-wise orthonormal. The $s_{\text{gen}}$ are the $n$-mode singular values of the tensor $Y$.

From these SVDs, we derive an approximation $Y_{\text{gen}}^{(1)}(x)$ of $Y$ by discarding the smallest $s$ n-mode singular values. i.e. by setting the corresponding parts of $C$ to zero. Then we have

$$\|Y - Y_{\text{gen}}^{(1)}(x)\|_F \leq \sum_{s=n+1}^{p} s_{\text{gen}}^2 \|x\|_F,$$

Numerical tests

We consider the Burgers equation with the viscosity parameter $\mu$

$$\partial_t u + u \partial_x u + \partial_x^2 u = 0,$$

with the spatial coordinate $x \in (0, 1)$, the time variable $t \in (0, 1)$, completed by zero Dirichlet boundary conditions and a step function as initial conditions as illustrated in Figure 2(a).

Assembling the measurement matrices

The spatial discretization is done through piecewise linear finite elements on an equidistant grid of $q$ nodes. For fixed choices of $\mu$, the solution trajectories are obtained via a Runge-Kutta solver and then tested against the basis functions of a $S \in L^2(0, 1)$ chosen as the span of $s$ equidistantly distributed linear hat functions.

Test setups

We use the parameter values $\mu_k = 10^{-2}, \mu_j = 3, \mu_{j-1} = 10^{-1}$ to set up the measurement tensor $Y$ and to compute the space and time POD modes. These POD modes are then used in a space-time Galerkin scheme for Equation 4. Thus the solution of the reduced model is obtained via the solution of a nonlinear equation system with $s \times q$ degrees of freedom. As the error measure, we use the space-time $L^2$ difference between a solution of the full and the reduced model.

![Figure 1: Burger setup for $\mu = 3 \times 10^{-2}$](image)

The full solution, the reduced solution, and the approximation error

Space vs. time resolution

We set the overall number of POD modes to $K = q \times s$ and consider various space time resolutions $q = f \times K$ and $s = 1 \ldots K$, for $f \in [0.2, 0.8]$. Examining the time-space approximation vs. $f$, one sees that $f = 0.5$, $a = q = s$ seems the best choice over the whole parameter range, cf. Figure 2(a).

Approximation error vs. parameter

We investigate the error for reduced systems of order $K = (20, 30, 40)$ in a parameter range within and slightly outside the training sets, see Figure 2(b).

Implementation

The code is available from the author’s public git repository [8].

![Figure 2: (a) the error for various numbers of $f$. (b) the error in the reduced model over the parameter range for various $K$.](image)