A generalization of the Proper Orthogonal Decomposition method for nonlinear model-order reduction

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Discrete input-output maps

An abstract input-output (I/O) map is given by,

\[ G : \mathcal{U} \rightarrow \mathcal{Y}, \quad u = u(t, \theta) \rightarrow y = y(t, \xi), \]

and can be discretized & reduced in the following three steps:

1. Assume space-time tensor structure for input and output spaces: \( \mathcal{U} = \mathcal{R}_{r_1} \otimes \mathcal{U}_{h_1} \) and \( \mathcal{Y} = \mathcal{S}_{r_2} \otimes \mathcal{Y}_{h_2} \), with \( \text{dim}(\mathcal{U}) = rp \) and \( \text{dim}(\mathcal{Y}) = sq \).
2. Define sets of bases such as \( \{ \psi_0, ..., \psi_s \} \) for \( \mathcal{S}_{r_2} \) with scalar product \( \langle \cdot, \cdot \rangle \).
3. By testing the I/O behavior, we obtain a tensor \( G \in \mathbb{R}^{r \times p \times q \times q} \) which can be unfolded and reduced via a higher-order SVD.

Classical POD vs. generalized POD

We consider a nonlinear dynamical system of the form,

\[ \dot{x}(t) = f(x(t), t), \quad x(0) = x_0, \quad x(t) \in \mathbb{R}^q, \]  \hspace{1cm} (1)

with output \( y = x \) and one-dimensional input (i.e. \( r = p = 1 \)).

Collect snapshot matrices at \( s \) time instances:

\[ X := \begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_s) \\ \vdots & \vdots & \ddots & \vdots \\ x_q(t_1) & x_q(t_2) & \cdots & x_q(t_s) \end{bmatrix} \leftrightarrow \begin{bmatrix} (x_1, \psi_1)x & \cdots & (x_1, \psi_s)x \\ \vdots & \ddots & \vdots \\ (x_q, \psi_1)x & \cdots & (x_q, \psi_s)x \end{bmatrix} \]

Define \( k \)-dimensional reduced-order model of (1),

\[ \hat{\dot{x}}(t) = U_k^T f(U_k \hat{x}(t), t), \quad \hat{x}(t) \in \mathbb{R}^k, \quad k < q, \]

where \( U_k \) consists of the \( k \)-leading left singular vectors of \( X \) (for classical POD) or of \( X_{gm}, M_S^{-1/2} \) (for gmPOD), and \( x(t) \approx U_k \hat{x}(t) \).

The reduction error \( e_{x,k} \) is measured by

\[ e_{x,k} := \left( \int_0^T \| x(t) - U_k \hat{x}(t) \|_{L_2(x(t))}^2 dt \right)^{1/2} \]

Example: Burgers’ equation

Fig. 3: Full-order model (left) and reduction errors of POD (middle) and gmPOD (right).

Numerical results

Fig. 4: Accuracy of POD vs. gmPOD.

References
