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A generalization of the Proper Orthogonal Decomposition method for nonlinear model-order reduction

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Discrete input-output maps

An abstract **input-output (I/O) map** is given by,

$$\mathbb{G} : \mathcal{U} \to \mathcal{Y}, \quad u = u(t, \theta) \mapsto y = y(t, \xi),$$

and can be discretized & reduced in the following three steps:

1. Assume space-time tensor structure for input and output spaces: $\mathcal{U} = \mathcal{R}_{\tau_1} \otimes U_{h_1}$ and $\mathcal{Y} = \mathcal{S}_{\tau_2} \otimes Y_{h_2}$, with $\dim(\mathcal{U}) = rp$ and $\dim(\mathcal{Y}) = sq$.
2. Define sets of bases such as $\{\psi_1, ..., \psi_s\}$ for $\mathcal{S}_{\tau_2}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{S}}$.
3. By testing the I/O behavior, we obtain a tensor $\mathbf{G} \in \mathbb{R}^{r \times p \times s \times q}$ which can be unfolded and reduced via a **higher-order SVD**.

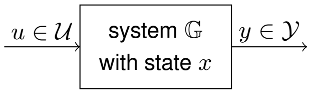

Fig. 1: An abstract I/O map with input u, state variable x, and output y.

Hierarchical basis for $\mathcal{S}_{\tau_2}$

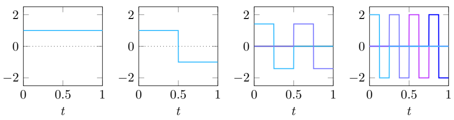

Fig. 2: Set of Haar wavelets $\{\psi_1, ..., \psi_8\}$.

Classical POD vs. generalized POD

We consider a nonlinear **dynamical system** of the form,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t) \in \mathbb{R}^q, \qquad (1)$$

with output $\mathbf{y} = \mathbf{x}$ and one-dimensional input (i.e. $r = p \equiv 1$).

Collect snapshot matrices at s time instances:

$$X := \begin{bmatrix} x_1(t_1) & \dots & x_1(t_s) \\ \vdots & \ddots & \vdots \\ x_q(t_1) & \dots & x_q(t_s) \end{bmatrix} \quad \leftrightarrow \quad X_{gm} := \begin{bmatrix} \langle x_1, \psi_1 \rangle_{\mathcal{S}} & \dots & \langle x_1, \psi_s \rangle_{\mathcal{S}} \\ \vdots & \ddots & \vdots \\ \langle x_q, \psi_1 \rangle_{\mathcal{S}} & \dots & \langle x_q, \psi_s \rangle_{\mathcal{S}} \end{bmatrix}$$

Define k-dimensional **reduced-order model** of (1),

$$\dot{\tilde{\mathbf{x}}}(t) = U_k^{\mathsf{T}} \mathbf{f}(U_k \tilde{\mathbf{x}}(t), t), \quad \tilde{\mathbf{x}}(t) \in \mathbb{R}^k, \quad k \ll q,$$

where U_k consists of the k-leading left singular vectors of X (for classical POD) or of $X_{gm} M_{\mathcal{S}}^{-1/2}$ (for gmPOD), and $\mathbf{x}(t) \approx U_k \tilde{\mathbf{x}}(t)$.

The **reduction error** $e_{s,k}$ is measured by

$$e_{s,k} := \left(\int_0^T \|\mathbf{x}(t) - U_k \tilde{\mathbf{x}}(t)\|^2_{L^2(\Omega)} dt \right)^{1/2}$$

Lemma

The $L^2(0,T)$-orthogonal projection $\tilde{\mathbf{x}}(t)$ of the state vector $\mathbf{x}(t)$ onto the space spanned by the measurements is given as

$$\tilde{\mathbf{x}}(t) = X_{gm} M_{\mathcal{S}}^{-1} \boldsymbol{\psi}(t),$$

where $\boldsymbol{\psi} := [\psi_1, ..., \psi_s]^{\mathsf{T}}$, and where $[M_{\mathcal{S}}]_{i,j} := \langle \psi_i, \psi_j \rangle_{\mathcal{S}}$.

The generalized measurements POD (gmPOD) basis can be computed via a **truncated SVD** of the matrix

$$X_{gm} M_{\mathcal{S}}^{-1/2}.$$

Proof: Can be found in [1].

Example: Burgers' equation

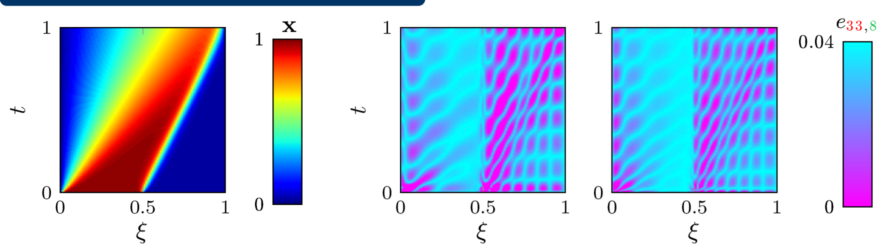

Fig. 3: Full-order model (left) and reduction errors of POD (middle) and gmPOD (right).

Numerical results

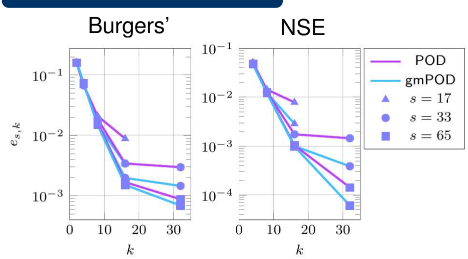

Fig. 4: Accuracy of POD vs. gmPOD.

References

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[2] M. Baumann. *Nonlinear model-order reduction using POD/DEIM for optimal control of Burgers' equation.* Master's thesis, Delft University of Technology, 2013. Scientific supervision by M. Rojas.