Model Order Reduction by Balanced Truncation

M. Ambrožič, M. Baumann, C. Echeverría Serur

November 26, 2012

Delft University of Technology
COSSE

Table of Contents

The overall goal of Model Order Reduction

Consider two systems

$$\dot{x} = Ax + Bu, \qquad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u,$$
$$y = Cx + Du, \qquad \tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u,$$

where $x \in \mathbb{R}^n, \tilde{x} \in \mathbb{R}^k$, with $k \ll n$.

We are aiming for an approximation such that

$$\|y - \tilde{y}\|_{\mathcal{L}_2}$$

is small.

Definition

The space $\mathcal{H}_2 := \{\hat{f} : \mathbb{C}^+ \to \mathbb{C} \mid \hat{f} \text{ holomorphic}, \|\hat{f}\|^2_{\mathcal{H}_2} < \infty\}$ is called **Hardy space**, where $\|\hat{f}\|^2_{\mathcal{H}_2} := \frac{1}{2\pi} \sup_{\sigma>0} \int_{-\infty}^{\infty} \|\hat{f}(\sigma + i\omega)\|_2^2 \, d\omega$.

For the transfer functions

$$G(s) = D + C(sI_n - A)^{-1}B,$$
$$\tilde{G}(s) = \tilde{D} + \tilde{C}(sI_k - \tilde{A})^{-1}\tilde{B},$$

the following holds

$$\|y - \tilde{y}\|_{\mathcal{L}_2} = \|\hat{y} - \hat{\tilde{y}}\|_{\mathcal{H}_2} = \|G\hat{u} - \tilde{G}\hat{u}\|_{\mathcal{H}_2} = \|(G - \tilde{G})\hat{u}\|_{\mathcal{H}_2}$$
$$\leq \|G - \tilde{G}\|_{\mathcal{H}_\infty}\|\hat{u}\|_{\mathcal{H}_2} = \|G - \tilde{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{L}_2},$$

where $\|G\|_{\mathcal{H}_\infty} := \sup_{\hat{u}\in\mathcal{H}_2\setminus\{0\}} \frac{\|\hat{y}\|_{\mathcal{H}_2}}{\|\hat{u}\|_{\mathcal{H}_2}}$.

Definition

The space $\mathcal{H}_2 := \{\hat{f} : \mathbb{C}^+ \to \mathbb{C} \mid \hat{f} \text{ holomorphic}, \|\hat{f}\|^2_{\mathcal{H}_2} < \infty\}$ is called **Hardy space**, where $\|\hat{f}\|^2_{\mathcal{H}_2} := \frac{1}{2\pi} \sup_{\sigma > 0} \int_{-\infty}^{\infty} \|\hat{f}(\sigma + i\omega)\|_2^2 \, d\omega$.

For the transfer functions

$$G(s) = D + C(sI_n - A)^{-1}B,$$
$$\tilde{G}(s) = \tilde{D} + \tilde{C}(sI_k - \tilde{A})^{-1}\tilde{B},$$

the following holds

$$\begin{aligned} \|y - \tilde{y}\|_{\mathcal{L}_2} &= \|\hat{y} - \hat{\tilde{y}}\|_{\mathcal{H}_2} = \|G\hat{u} - \tilde{G}\hat{u}\|_{\mathcal{H}_2} = \|(G - \tilde{G})\hat{u}\|_{\mathcal{H}_2} \\ &\leq \|G - \tilde{G}\|_{\mathcal{H}_\infty} \|\hat{u}\|_{\mathcal{H}_2} = \|G - \tilde{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}, \end{aligned}$$

where $\|G\|_{\mathcal{H}_\infty} := \sup_{\hat{u} \in \mathcal{H}_2 \setminus \{0\}} \frac{\|\hat{y}\|_{\mathcal{H}_2}}{\|\hat{u}\|_{\mathcal{H}_2}}$.

Balanced truncation is based on two ideas

1. **Balance** the full order system (A, B, C, D), i.e. transform the system such that its Gramians are *equal* and *diagonal*.
2. **Truncate** the balanced system with a small error $\|G - \tilde{G}\|_{\mathcal{H}_\infty}$.
3. Preserve nice properties of the original system if possible.

Definition

The **controllability Gramian** P and the **observability Gramian** Q of the system (A, B, C, D) can be uniquely defined by the solutions of the two Lyapunov equations

$$AP + PA^T = -BB^T,$$
$$A^T Q + QA = -C^T C.$$

Definition

A system (A, B, C, D) is called **balanced** if the Gramian P and the Gramian Q are equal, positive definite and have descending diagonal elements, i.e. $P = Q = \Sigma$ with

$$\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n), \qquad \text{where } \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n > 0.$$

The numbers $\sigma_1, ..., \sigma_s$ are the so-called **Hankel singular values**.

Idea of MOR: We want to eliminate states $\tilde{x}$ that are

- *"hard to reach"*

$$\tilde{x}^T P^{-1} \tilde{x} = \min \left\{ \|u\|^2_{\mathcal{L}_2} \;\middle|\; \begin{array}{l} u \text{ solves } \dot{x} = Ax + Bu, \\ x(0) = 0, x(t_f) = \tilde{x} \end{array} \right\} \gg 1$$

- and *"hard to observe"*

$$\tilde{x}^T Q \tilde{x} = \{ \|y\|^2_{\mathcal{L}_2} \mid y \text{ fulfills } y = Cx, \dot{x} = Ax, x(0) = \tilde{x} \} \ll 1$$

The gain of balancing

Balancing the system (A, B, C, D) will be useful to determine states that fulfill both of the above conditions.

Idea of MOR: We want to eliminate states $\tilde{x}$ that are

- *"hard to reach"*

$$\tilde{x}^T P^{-1} \tilde{x} = \min \left\{ \|u\|^2_{\mathcal{L}_2} \;\middle|\; \begin{array}{l} u \text{ solves } \dot{x} = Ax + Bu, \\ x(0) = 0, x(t_f) = \tilde{x} \end{array} \right\} \gg 1$$

- and *"hard to observe"*

$$\tilde{x}^T Q \tilde{x} = \{ \|y\|^2_{\mathcal{L}_2} \mid y \text{ fulfills } y = Cx, \dot{x} = Ax, x(0) = \tilde{x} \} \ll 1$$

The gain of balancing

Balancing the system (A, B, C, D) will be useful to determine states that fulfill both of the above conditions.

Lemma

Let $P, Q \in \mathbb{R}^{n,n}$ with $P, Q \geq 0$. Then there exists a similarity transformation S, such that

$$S^{-1}AS = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{34} & A_{44} \end{bmatrix}, \; S^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \; CS = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}$$

and

$$S^{-1}PS^{-T} = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad S^T QS = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix},$$

with $\Sigma_1, \Sigma_2, \Sigma_3$ diagonal and SPD.

Note: (A_{11}, B_1, C_1, D) is a **balanced** realization of (A, B, C, D).

Proof. 1/2

Consider the Cholesky decomposition of the Gramians

$$P = WW^T, \quad Q = RR^T$$

and the singular value decomposition (SVD)

$$W^T R = U \Sigma V^T, \quad \text{with } \Sigma = diag(\sigma_1, ..., \sigma_n).$$

Define the transformation $S := WU\Sigma^{-1/2}$.

Then it holds

$$(\Sigma^{-1/2} V^T R^T) S = (\Sigma^{-1/2} V^T R^T)(WU\Sigma^{-1/2})$$
$$= \Sigma^{-1/2} V^T V \Sigma U^T U \Sigma^{-1/2} = I_n,$$

i.e. $S^{-1} = \Sigma^{-1/2} V^T R^T$.

Proof. 1/2

Consider the Cholesky decomposition of the Gramians

$$P = WW^T, \quad Q = RR^T$$

and the singular value decomposition (SVD)

$$W^T R = U \Sigma V^T, \quad \text{with } \Sigma = diag(\sigma_1, ..., \sigma_n).$$

Define the transformation $S := WU\Sigma^{-1/2}$.
Then it holds

$$\begin{aligned}(\Sigma^{-1/2} V^T R^T) S &= (\Sigma^{-1/2} V^T R^T)(WU\Sigma^{-1/2}) \\ &= \Sigma^{-1/2} V^T V \Sigma U^T U \Sigma^{-1/2} = I_n,\end{aligned}$$

i.e. $S^{-1} = \Sigma^{-1/2} V^T R^T$.

Proof. 2/2

The transformed system

$$A_b = S^{-1}AS, \; B_b = S^{-1}B, \; C_b = CS$$

is balanced because

$$\begin{aligned} P_b &= S^{-1}PS^{-T} \\ &= \Sigma^{-1/2}V^T R^T WW^T RV\Sigma^{-1/2} \\ &= \Sigma^{-1/2}V^T V\Sigma U^T U\Sigma V^T V\Sigma^{-1/2} = \Sigma, \end{aligned}$$

$$\begin{aligned} Q_b &= S^T QS \\ &= \Sigma^{-1/2}U^T W^T RR^T WU\Sigma^{-1/2} \\ &= \Sigma^{-1/2}U^T U\Sigma V^T V\Sigma U^T U\Sigma^{-1/2} = \Sigma. \end{aligned}$$

□

The reduced system

Assume (A_{11}, B_1, C_1, D) is a balanced realization with Gramians $P = Q = \Sigma_1 = diag(\sigma_1, ..., \sigma_n)$, where $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n > 0$.
Consider the partition

$$A_{11} = \begin{bmatrix} \tilde{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B_1 = \begin{bmatrix} \tilde{B} \\ B_2 \end{bmatrix}, \; C_1 = \begin{bmatrix} \tilde{C} & C_2 \end{bmatrix}.$$

The reduced system is given by $(\tilde{A}, \tilde{B}, \tilde{C}, D)$ with Gramians $\tilde{P} = \tilde{Q} = diag(\sigma_1, ..., \sigma_r)$ and transfer function $\tilde{G}$.

The reduced system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u,$$
$$\tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u,$$

has the following properties:

1. The following error bound holds $\|G - \tilde{G}\|_{\mathcal{H}_\infty} \leq 2\sum_{k=r+1}^{n} \sigma_k$,
2. stability is preserved.

Note that property 1 is nice because of

$$\|y - \tilde{y}\|_{\mathcal{L}_2} \leq \|G - \tilde{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}$$

The reduced system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u,$$
$$\tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u,$$

has the following properties:

1. The following error bound holds $\|G - \tilde{G}\|_{\mathcal{H}_\infty} \leq 2\sum_{k=r+1}^{n} \sigma_k$,
2. stability is preserved.

Note that property 1 is nice because of

$$\|y - \tilde{y}\|_{\mathcal{L}_2} \leq \|G - \tilde{G}\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}$$

Stability is preserved

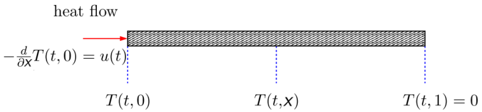
Theorem

Given a stable full order system (A, B, C, D), then the reduced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ obtained by balanced truncation has the following properties:

1. *$\tilde{A}$ has no eigenvalues in the open right half plane, i.e. the reduced system is **stable**.*
2. *If, additionally, $\sigma_i \neq \sigma_j$, for $i = 1, \ldots, r, j = r+1, \ldots, n$, then $\tilde{A}$ has no eigenvalues on $i\mathbb{R}$, i.e. the reduced system is **asymptotically stable**.*

A numerical Example

Consider the following control problem



for the heat equation

$$\frac{\partial}{\partial t} T(t,x) = k \cdot \frac{\partial^2}{\partial x^2} T(t,x)$$

with boundary conditions

$$-\frac{\partial}{\partial x} T(t,0) = u(t), \quad T(t,1) = 0,$$

and output

$$y(t) := \int_0^1 T(t,x)\, dx.$$

The temperature is considered as *state* of the system.

Spacial discretization (MoL) leads to:

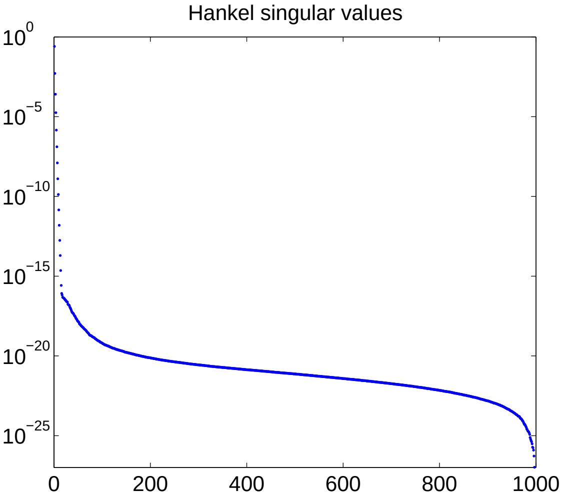
$$\begin{aligned} \dot{T}(t) &= AT(t) + Bu(t), \;\; T(0) = 0, \\ y(t) &= CT(t), \end{aligned}$$

with

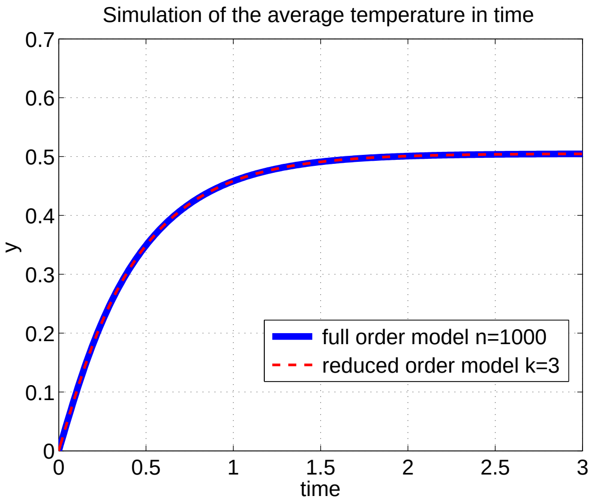
$$A = k \cdot n^2 \cdot \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{n,n}, \quad B = k \cdot n \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n,1},$$

$$C = \frac{1}{n} \cdot \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

Decay of the Hankel singular values



Numerical solution



Summary

- Balanced truncation is a model order reduction algorithm with guaranteed error bound on the output approximation.
- In our example, an ODE system of dimension $n = 1000$ was reduced by balanced truncation to a system of dimension $k = 3$ with an error bound of $\|G - \tilde{G}\|_{\mathcal{H}_\infty} \sim \mathcal{O}(10^{-5})$.
- For very large dimension n, the balanced truncation method is no longer computational feasible due to the SVD which has complexity $\mathcal{O}(n^3)$.

Summary

- Balanced truncation is a model order reduction algorithm with guaranteed error bound on the output approximation.
- In our example, an ODE system of dimension $n = 1000$ was reduced by balanced truncation to a system of dimension $k = 3$ with an error bound of $\|G - \tilde{G}\|_{\mathcal{H}_\infty} \sim \mathcal{O}(10^{-5})$.
- For very large dimension n, the balanced truncation method is no longer computational feasible due to the SVD which has complexity $\mathcal{O}(n^3)$.

Summary

- Balanced truncation is a model order reduction algorithm with guaranteed error bound on the output approximation.
- In our example, an ODE system of dimension $n = 1000$ was reduced by balanced truncation to a system of dimension $k = 3$ with an error bound of $\|G - \tilde{G}\|_{\mathcal{H}_\infty} \sim \mathcal{O}(10^{-5})$.
- For very large dimension n, the balanced truncation method is no longer computational feasible due to the SVD which has complexity $\mathcal{O}(n^3)$.