The invariant density of a chaotic dynamical system with small noise

- Stochastic Differential Equations -

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May 15, 2012

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Introduction

The stochastic (or chaotic) dynamical system

We consider

$$d\mathbf{x} = \mathbf{U}(\mathbf{x})dt + \sqrt{2}\epsilon\sigma d\mathbf{W}, \quad \mathbf{x}_0 \text{ given},$$

and, more precisely,

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \mu x - y^2 + 2z^2 - \delta z \\ y(x-1) \\ \mu z + \delta x - 2xz \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2}\epsilon \\ 0 \end{pmatrix} dW$$

Here, dW is white noise and the parameter ϵ determines the noise level. Note, that $\epsilon = 0$ gives a deterministic ODE.

Some **facts** about stochastic dynamical systems:

- Around 1900, H. Poincaré considered the orbits arising from *sets* of initial points.
- *Chaos* in real physical systems was not widely appreciated.
- Nowadays, numerical solution of dynamical systems *proves* chaotic behavior.

Chaotic dynamical systems occur in plenty **applications**:

1. shear instability of tall thin convection cells
2. laser oscillations consisting of short pulses separated by long periods of very small intensity

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For the case $\epsilon = 0$, we get a deterministic, ordinary differential equation - known as **dynamical system**

$$\mathbf{x}' = \mathbf{U}(\mathbf{x}),$$

which can be solved by linearization and analysis of **critical points**.

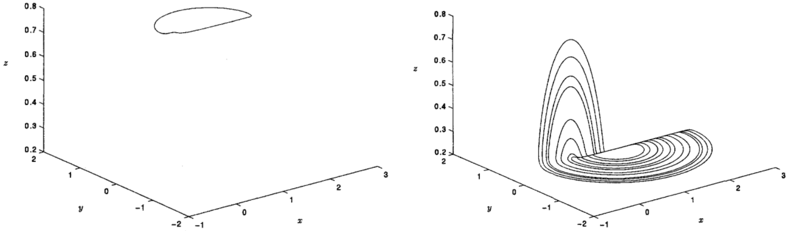
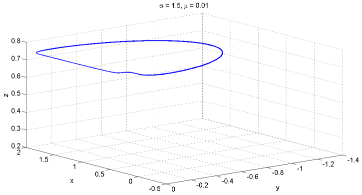
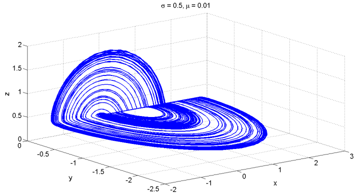


Figure: Phase Solution for $\delta = 1.5, \mu = 0.01$ (left) and $\delta = 0.5, \mu = 0.01$ (right).

We tried to re-compute the results from the paper by a *simple* Matlab implementation.





- forward Euler discretization
- solution trajectory is periodic for the case $\delta = 1.5$

For $\epsilon > 0$, the system becomes stochastic.

- noise is added to the y-component
- avoids hitting the unstable critical point at $y = 0$
- suppresses large x and z excursions

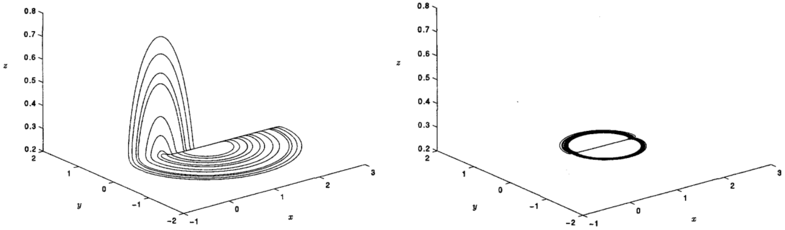


Figure: Deterministic and stochastic dynamical system

Considering a SDE of the form

$$dx = f_1(x,y,z)dt, \;\; dy = f_2(x,y,z)dt + \sqrt{2}\epsilon dW, \;\; dz = f_3(x,y,z)dt.$$

The Fokker-Planck equation (FPE)

The transition probability density $p(t,x,y,z)$ satisfies the (FPE)

$$\epsilon^2 p_{yy} - (f_1 p)_x - (f_2 p)_y - (f_3 p)_z \equiv \mathcal{L}^* p = p_t$$

The **invariant** probability density p is given by the stationary solution of this equation, which is

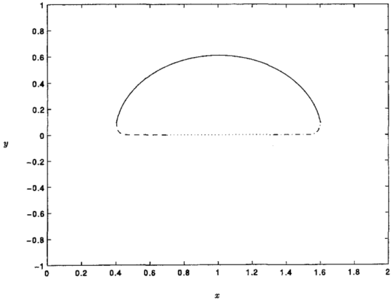
$$\mathcal{L}^* p = 0 \quad \Leftrightarrow \quad p_t = 0.$$

In the paper, the following form of the invariant density has been derived

$$p(x, y) \sim K(x) e^{-(y - F(x) + \epsilon/\sqrt{g_n(x)})^2 g(x)/(2\epsilon^2)}$$

The distribution is centered around the curve

$$y = F(x) - \epsilon/\sqrt{g_n(x)}$$



Summary and Conclusions

- An **analytic** expression for the invariant probability density of the considered system has been derived.
- Therefore, the effect of small noise to the system is well understood.
- Suppressing of large x and z excursions in the stochastic case ("noisy periodicity").

Thanks for your Attention!

Are there any questions / remarks ?

R. Kuske, G. Papanicolaou. *The invariant density of a chaotic dynamical system with small noise.* Elsevier Science B.V. Physica D 120 (1998) 255-272